Counterexamples of the Geometrization Conjecture

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Abstract

In this paper we propose counterexamples to the Geometrization Conjecture and the Elliptization Conjecture.

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1 A counterexample of the Geometrization Conjecture

A version of the Thurston's Geometrization Conjecture states that if a closed (oriented and connected) 3-manifold is irreducible and atoroidal, then it is geometric in the sense that it can either have a hyperbolic geometry or have a spherical geometry [1][2][3]. In this paper we propose counterexamples to this conjecture by using the Dehn surgery method of constructing closed 3-manifolds [4][5].

Let K^1_{RT} denote the right trefoil knot with framing 1. Let K^r_E denote the figure-eight knot with framing r where $r = \frac{p}{q}$ is a rational number (p and q are co-prime integers) such that $r > 4$. We then consider a Dehn surgery on the framed link $L = K^1_{RT} \cup K^r_E$ where the linking $\cup$ is of the simplest Hopf link type.

We have that the Dehn surgery on K^1_{RT} gives the Poincaré sphere M^1_{RT} which is with spherical geometry and with a finite nontrivial fundamental group [1][2][4][6][7]. Then the Dehn surgery on K^r_E gives a hyperbolic manifold M^r_E [1][2][6][7]. We want to show that the 3-manifold M_L obtained from surgery on L is irreducible and atoroidal, and is not geometric. From this we then have that M_L is a counterexample of the Geometrization Conjecture.

Let us first show that M_L is irreducible and atoroidal. From [9] we have the following quantum invariant $\overline{W}(K^1_{RT})$ of M^1_{RT}:

$$\overline{W}(K^1_{RT}) = R^2 R_1^{-1} R_2^1 W(C_1) W(C_2) \tag{1}$$

where the indexes of the R-matrices R_1 and R_2 are 1 and -1 respectively (These R-matrices are the monodromies of the Knizhnik-Zamolodchikov equation; the notation $W(K)$ denotes the generalized Wilson loop of a knot K and is a quantum representation of K [9]). Thus the indexes of R_1 and R_2 are nonzero and are different. In [9] we call this property as the maximal non-degenerate property which is a property only from nontrivial knots. We have that R_1 and R_2 act on $W(C_1)$ and $W(C_2)$ respectively while R is a R-matrix for the linking of the framed knot K^1_{RT} and acts on $W(C_1)$ and $W(C_2)$. Similarly we have the following quantum invariant of M^r_E:

$$\overline{W}(K^r_E) = R^{2p} R_1^{-3} R_2^{-a3} W(C_1) W(C_2) \tag{2}$$

where we choose a rational number $r = \frac{p}{q}$ such that the integer $a \neq 1$ is nonzero. This is then the maximal non-degenerate property.

Now let us consider the manifold M_L. Since K^1_{RT} and K^r_E both have the maximal non-degenerate property we have that there is no degenerate degree of freedom for the quantum representation of M_L by using the link L. From this we have that L is a minimal link for the

Dehn surgeries obtaining M_L [9] (We shall later give more explanations on the definition of minimal link and the related theorems on the classification of 3-manifolds by quantum invariant of 3-manifolds). It follows that the quantum invariant of M_L is given by the quantum representation of L and is of the following form:

$$\overline{W}(L) = P_L \overline{W}(K^1_{RT})\overline{W}(K^r_E) \tag{3}$$

where P_L denotes the linking part of the representation of L.

In this quantum invariant (3) of M_L we have that $\overline{W}(K^1_{RT})$ and $\overline{W}(K^r_E)$ representing K^1_{RT} and K^r_E respectively are independent of each other and that the framed knots K^1_{RT} and K^r_E are independent of each other in the sense that the framed knots K^1_{RT} and K^r_E do not wind each other in the form as described by the second Kirby move [4][8].

We have that the quantum invariant (3) of M_L uniquely represents M_L because L is minimal (We shall explain this point in the next section). This means that there are no nontrivial symmetry transforming it to another representation of M_L with two framed knots such that their quantum representations are different from the two quantum representations $\overline{W}(K^1_{RT})$ and $\overline{W}(K^r_E)$ in (3).

Let us then first show that M_L is irreducible. Since the quantum invariant (3) of M_L uniquely represents M_L and thus represents topological properties of M_L we have that the linking part P_L of (3) is a topological property of M_L and thus cannot be eliminated. From this linking of $\overline{W}(K^1_{RT})$ and $\overline{W}(K^r_E)$ in (3) we have that the invariant (3) of M_L cannot be written as a free product form $\overline{W}(K^{r_1}_1)\overline{W}(K^{r_2}_2)$ of two unlinked framed knots $K^{r_1}_1$ and $K^{r_2}_2$ where each $\overline{W}(K^{r_i}_i), i = 1, 2$ gives a closed 3-manifold. From this we have that M_L cannot be written as a connected sum of two closed 3-manifolds. This shows that M_L is irreducible.

Then we want to show that M_L is atoroidal. Since the toroidal property of a 3-manifold M is about the existence of an infinite cyclic subgroup $Z \oplus Z$ in $\pi_1(M)$ and is a property derived from closed curves in M only we have that this toroidal property is derived from framed knots only since framed knots are closed curves for constructing 3-manifolds. Now since L is minimal we have that the representation (3) uniquely represents M_L and thus it gives all the topological properties of M_L. From this we have that if M_L has the toroidal property then this property can only be derived from the two framed knot components K^1_{RT} and K^r_E. Now we have that the 3-manifolds M^1_{RT} and M^r_E are both atoroidal and that the fundamental group of M^1_{RT} is finite [1][2][6][7]. Thus the two framed knot components K^1_{RT} and K^r_E do not give the toroidal property of M_L. This shows that M_L does not have the toroidal property. Thus M_L is atoroidal.

Let us explicitly compute the fundamental group $\pi_1(M_L)$ of M_L to give another proof for that M_L is atoroidal. We have that $L = K^1_{RT} \cup K^r_E$ is of the Hopf link type. Thus by a computation similar to the computation of the link group of the Hopf link which is a direct product of the two knot groups of the two unknots forming the Hopf link we have that the fundamental group $\pi_1(M_L)$ of M_L is a direct product of the fundamental groups $\pi_1(M^1_{RT})$ and $\pi_1(M^r_E)$:

$$\pi_1(M_L) = \pi_1(M^1_{RT}) * \pi_1(M^r_E) \tag{4}$$

where $\pi_1(M^1_{RT}) * \pi_1(M^r_E)$ denotes the direct product of the fundamental groups $\pi_1(M^1_{RT})$ and $\pi_1(M^r_E)$. Now since the 3-manifolds M^1_{RT} and M^r_E are both atoroidal and that the fundamental group $\pi_1(M^1_{RT})$ is finite we have that $\pi_1(M_L)$ does not contain a subgroup of the form $Z \oplus Z$. This shows that M_L does not have the toroidal property. Thus M_L is atoroidal.

Now since the quantum invariant (3) uniquely represents M_L we have that the two components $\overline{W}(K^1_{RT})$ and $\overline{W}(K^r_E)$ are topological properties of M_L. Then since $\overline{W}(K^1_{RT})$ (or K^1_{RT}) gives spherical geometry property to M_L and $\overline{W}(K^r_E)$ (or K^r_E) gives hyperbolic geometry property to M_L we have that M_L is not geometric. Indeed, since the two independent components $\overline{W}(K^1_{RT})$ and $\overline{W}(K^r_E)$ of (3) represent the manifolds M_{RT} and M_E respectively (and thus represent the fundamental groups $\pi_1(M_{RT})$ and $\pi_1(M_E)$ of M_{RT} and M_E respectively) we have that the fundamental group $\pi_1(M_L)$ of M_L contains the direct product $\pi_1(M_{RT}) * \pi_1(M_E)$ of the fundamental groups $\pi_1(M_{RT})$ and $\pi_1(M_E)$. Now let $\tilde{M}_L$ denote the universal covering space of M_L. Then we have that $\pi_1(M_L)$ acts isometrically on $\tilde{M}_L$. Now since $\pi_1(M_{RT})$ of the Poincaré sphere M_{RT} is not a

subgroup of the isometry group of the hyperbolic geometry H^3 and $\pi_1(M_E)$ is not a subgroup of the isometry group of the spherical geometry S^3 we have that $\pi_1(M_{RT})*\pi_1(M_E)$ is not a subgroup of the isometry group of H^3 and is not a subgroup of the isometry group of S^3. Thus $\pi_1(M_L)$ is not a subgroup of the isometry group of H^3 and is not a subgroup of the isometry group of S^3. It follows that $\tilde{M}_L$ is not the hyperbolic geometry H^3 and is not the spherical geometry S^3. This shows that M_L is not geometric, as was to be proved. Now since M_L is irreducible and atoroidal and is not geometric we have that M_L is a counterexample of the Geometrization Conjecture.

2 Minimal link and classification of closed 3-manifolds

In this section we give more explanations on the definition of minimal link and the related theorems on the classification of closed 3-manifolds by quantum invariant used in the above counterexample.

We have the following theorem of one-to-one representation of 3-manifolds obtained from framed knots $K^{\frac{p}{q}}$ [9]:

Theorem 1 *Let M be a closed (oriented and connected) 3-manifold which is constructed by a Dehn surgery on a framed knot $K^{\frac{p}{q}}$ where K is a nontrivial knot and M is not a lens space. Then we have the following one-to-one representation of M:*

$$\overline{W}(K^{\frac{p}{q}}) := R^{2p}R_1^{-m}R_2^{-am}W(C_1)W(C_2) \tag{5}$$

where $m \neq 0$ (m is also denoted by m_1 in [9]) is the index of a nontrivial knot (which may or may not be the knot K such that M is also obtained from this knot by Dehn surgery) and $am \neq 0$ is an integer related to m, p and q such that $am \neq m$ (Thus (5) is with the maximal non-degenerate property).

We remark that if M is a lens space we can also define a similar quantum invariant $\overline{W}(K^{\frac{p}{q}})$ for M which however is not of the above maximal non-degenerate form [9].

Let us then consider a 3-manifold M which is obtained from a framed link L with the minimal number n of component knots where $n \geq 2$ (where the minimal number n means that if M can also be obtained from another framed link then the number of component knots of this framed link must be $\geq n$) . In this case we call L a minimal link of M. From the generalized second Kirby moves (which generalizes second Kirby move from integer to rational number [9] and for simplicity we shall call them again as the second kirby moves) we may suppose that L is in the form that the components $K_i^{\frac{p_i}{q_i}}, i = 1, ..., n$ of L do not wind each other in the form described by the second Kirby move. In this case we say that this minimal L is in the form of maximal non-degenerate state where the degenerate property is from the winding of one component knot with the other component knot by the second Kirby moves. Thus this L has both the minimal and maximal property as described. Then we want to find a one-to-one representation (or invariant) of M from this L. Let us write $W(L)$, the generalized Wilson loop of L, in the following form [9]:

$$W(L) = P_L \prod_i W(K_i^{\frac{p_i}{q_i}}) \tag{6}$$

where P_L denotes a product of R-matrices acting on a subset of $\{W(K_i), W(K_{ic}), i = 1, ..., n\}$ where $W(K_i^{\frac{p_i}{q_i}})$ are independent (This is from the form of L that the component knots K_i are independent in the sense that they do not wind each other by the second Kirby moves). Then we consider the following representation (or invariant) of M:

$$\overline{W}(L) := P_L \prod_i \overline{W}(K_i^{\frac{p_i}{q_i}}) \tag{7}$$

where we define $\overline{W}(K_i^{\frac{p_i}{q_i}})$ by (5) and they are independent. We then have the following theorem:

Theorem 2 *Let M be a closed (oriented and connected) 3-manifold which is constructed by a Dehn surgery on a minimal link L with the minimal number n of component knots (and with the maximal property). Then we have that (7) is a one-to-one representation (or invariant) of M.*

Proof. We want to show that (7) is a one-to-one representation (or invariant) of M. Let L' be another framed link for M which is also with the minimal number n (and with the maximal property). Then we want to show $\overline{W}(L) = \overline{W}(L')$.

For the case $n = 1$ this is true by the above theorem for manifolds M obtained from minimal framed knot $K^{\frac{p}{q}}$.

Let us consider $n \geq 2$. Since the components of L do not wind each other as described by the second Kirby move we have that the components of L are independent of each other. Thus there is no nontrivial homeomorphism changing these components $\overline{W}(K_i^{\frac{p_i}{q_i}})$ except those homeomorphisms involving the second Kirby moves for the winding of the components of L with each other. Then under the second Kirby moves we have that the components of L wind each other and thus will reduce the independent degree of freedom to be less than n. Thus to restore the degree of freedom to n these homeomorphisms must also contain the first Kirby moves of adding unknots with framing ± 1. In this case these unknots can be deleted and thus L is not minimal and this is a contradiction. Thus there is no nontrivial homeomorphism changing the components $\overline{W}(K_i^{\frac{p_i}{q_i}})$ of $\overline{W}(L)$ except those homeomorphisms consist of only the second Kirby moves for the winding of the components of L with each other.

Now suppose that $\overline{W}(L) \neq \overline{W}(L')$. Then there exists nontrivial homeomorphism of changing L to L' for changing the components $\overline{W}(K_i^{\frac{p_i}{q_i}})$ of $\overline{W}(L)$ to the components of $\overline{W}(L')$. This is impossible since there are no nontrivial homeomorphsm for changing these components $\overline{W}(K_i^{\frac{p_i}{q_i}})$ except those homeomorphisms consist of only the second Kirby moves for the winding of the components of L with each other. Thus $\overline{W}(L) = \overline{W}(L')$.

Thus we have that (7) is a one-to-one representation (or invariant) of M, as was to be proved. $\diamond$

As a converse to the above theorem let us suppose that the representation (7) uniquely represents M_L in the sense that there are no nontrivial symmetry transforming the n independent components of $\overline{W}(L)$ to other n independent components of $\overline{W}(L')$ where the link L' also gives the manifold M_L. Then from the above proof we see that the link L is a minimal (and maximal) link for obtaining M_L.

Remark. Let L be a minimal (and maximal) framed link. Then from the above proof we have that the components of L are independent of each other in the sense that if we transform a component framed knot of L to an equivalent framed knot by a homeomorphism then the other components of L are not affected by this transformation. $\diamond$

Now let us consider the framed link $L = K_{RT}^1 \cup K_E^r$ in the above section. We have that the knot components K_{RT}^1 and K_E^r of L do not wind each other in the form as described by the second Kirby move. Thus we have that their corresponding quantum invariants $\overline{W}(K_{RT}^1)$ and $\overline{W}(K_E^r)$ are independent. Then $\overline{W}(K_{RT}^1)$ and $\overline{W}(K_E^r)$ are in the maximal non-degenerate form which is invariant under all homeomorphisms execept the second Kirby moves which are excluded (Indeed for $\overline{W}(K_{RT}^1)$ there is a homeomorphism transforming K_{RT}^1 to K_E^{-1}. Then the informations of these two frame knots are included in $\overline{W}(K_{RT}^1)$ and thus $\overline{W}(K_{RT}^1)$ is invariant under this homeomorphism. Then since $\overline{W}(K_{RT}^1)$ is in the maximal non-degenerate form there are no degenerate degree of freedoms for other homeomorphisms execept the second Kirby moves which reduce the degree of freedom of L. Similarly for $\overline{W}(K_E^r)$). Thus L is a minimal (and maximal) link of M_L and the representation (3) is the quantum invariant of M_L.

3 A counterexample of the Elliptization Conjecture

The above counterexample of the Geometrization Conjecture is with an infinite fundamental group. Let us in this section propose a counterexample which is with a finite fundamental group to the Geometrization Conjecture. This example is then also a counterexample of the Thurston's Elliptization Conjecture which states that if a closed (oriented and connected) 3-manifold is irreducible and atoroidal and is with a finite fundamental group then it is geometric in the sense that it can have a spherical geometry [1][2][3].

Let us consider a Dehn surgery on the framed link $L = K_{RT}^1 \cup K_{RT}^1$ where the linking $\cup$ is of the simplest Hopf link type. We want to show that the 3-manifold M_L obtained from this surgery is a counterexample of the Elliptization Conjecture.

As similar to the above example we have that this L is minimal and the 3-manifold M_L is uniquely represented by the following quantum invariant:

$$\overline{W}(L) = P_L \overline{W}(K_{RT}^1)\overline{W}(K_{RT}^1) \tag{8}$$

where P_L denotes the linking part of the representation of L.

Then as similar to the above example we have that this 3-manifold M_L is irreducible and atoroidal. Let us then show that M_L is with a finite fundamental group and is not geometric. Since the quantum invariant (8) uniquely represents M_L we have that the two components $\overline{W}(K_{RT}^1)$ are topological properties of M_L. Then we have that the fundamental group $\pi_1(M_L)$ of M_L contains the direct product $\pi_1(M_{RT}) * \pi_1(M_{RT})$.

Further as similar to the above example because L is of the Hopf link type we have that $\pi_1(M_L) = \pi_1(M_{RT}^1) * \pi_1(M_{RT}^1)$. Now since the fundamental group $\pi_1(M_{RT})$ is finite we have that the fundamental group $\pi_1(M_L)$ is also finite.

Now let $\tilde{M}_L$ denote the universal covering space of M_L. Then we have that $\pi_1(M_L)$ acts isometrically on $\tilde{M}_L$. We want to show that $\tilde{M}_L$ is not the 3-sphere S^3. Suppose this is not true. Then since $\pi_1(M_L)$ contains (and equals to) the direct product $\pi_1(M_{RT}) * \pi_1(M_{RT})$ we have that the direct product $\pi_1(M_{RT}) * \pi_1(M_{RT})$ is a subgroup of the isometry group of S^3. Now since S^3 is a fully isotropic manifold containing no boundary (S^3 is closed) there is no way to distinguish two identical but independent subgroups $\pi_1(M_{RT})$ of the isometry group of S^3. From this we have that the direct product $\pi_1(M_{RT}) * \pi_1(M_{RT})$ can only act on $S^3 \times S^3$ where each $\pi_1(M_{RT})$ acts on a different S^3 and cannot act on the same S^3 such that $\pi_1(M_{RT}) * \pi_1(M_{RT})$ acts on S^3 (Comparing to the hyperbolic case we have that the direct product of two subgroups of the isometry group of the hyperbolic geometry H^3 may act on H^3 since H^3 has nonempty boundary which can be used to distinguish two identical but independent subgroups of the isometry group of H^3). Thus the direct product $\pi_1(M_{RT}) * \pi_1(M_{RT})$ is not a subgroup of the isometry group of S^3 (We can also prove this statement by the fact that $\pi_1(M_{RT})$ is a nonabelian subgroup of the rotation group $O(4)$ which is the isometry group of S^3. Indeed since $\pi_1(M_{RT})$ is nonabelian it must act on a space with dimension ≥ 3. Thus $\pi_1(M_{RT}) * \pi_1(M_{RT})$ must act on a space with dimension ≥ 6. Now $O(4)$ can only act on a space with dimension 4 we have that $\pi_1(M_{RT}) * \pi_1(M_{RT})$ is not a subgroup of $O(4)$). This is a contradiction. This contradiction shows that $\tilde{M}_L$ is not the 3-sphere S^3. Thus M_L is not geometric. Now since M_L is irreducible and atoroidal and is with finite fundamental group and is not geometric we have that M_L is a counterexample of the Elliptization Conjecture.

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